The Energy-Momentum Tensor for Gravitational Interactions

Walter Wyss

Department of Physics
University of Colorado
Boulder, CO 80309

Abstract

Within the Lagrange formalism we show that the gauge invariant total energy-momentum tensor for gravitational interactions is zero. If the equations of motion are satisfied the energy tensor is conserved.

I. Introduction

All our considerations are within the Lagrange formalism of field theory. The concept of the energy-momentum tensor for gravitational interactions has a long history. In this paper we adopt the philosophy commonly accepted for electromagnetic interactions. The electromagnetic field is described by a restmass zero, spin one field. This field then has its own gauge group. A charged matter field also has its own gauge group, that leads to a conserved current. If we demand that the electromagnetic interaction with the charged matterfield is a minimal coupling to the conserved current, the two gauge groups get related to form the EM interaction gauge group (electromagnetic gauge group). It consists of scalar functions and gives rise to a covariant derivative. The total action is then Lorentz-invariant and also EM gauge invariant. Similarly, the gravitational field is described by a restmass zero, spin two field. This field also has its own gauge group. If we demand that the gravitational interaction with a matterfield is a minimal coupling to the conserved energy-momentum tensor of the matterfield, the gauge group gets enlarged to the G interaction gauge group. Its Lie algebra consists of vector fields and gives raise to a covariant derivative. The total action is then Lorentz invariant and also G-gauge invariant [2] (gravitational gauge group).

We take the position that the Lorentz group is the fundamental symmetry group of all of physics. If there are additional symmetry groups, the physical relevant quantities should be covariant with respect to the additional symmetry. Conservation laws can involve quantities that are not covariant with respect to the additional symmetry groups but are only Lorentz covariant.

In this paper we study gravitational interactions with a general matter field. The G interaction gauge group consists of vector fields vanishing at infinity. The Lorentz group is thus not a subgroup of our G gauge group. With respect to the Lorentz group, the energy tensor (translation invariance) and the energy-momentum tensor (proper Lorentz invariance) are of physical importance [3]. We show that the energy tensor satisfies a conservation law and the G gauge invariant energy-momentum tensor vanishes. G gauge invariance implies additional conservation laws.

II. Gravitational Interaction with a General Matterfield

Let $g_{\alpha\beta}$ be a symmetric Lorentz tensor field on Minkowski space and $g^{\alpha\beta}$ its inverse. ϕ stands for a general multicomponent field on Minkowski space. We assume throughout automatic summation over Lorentz-, spinor-, and internal indices even if they are not explicitly mentioned. The action for gravitational interaction is given by

$$A = \int dx L \tag{II.1}$$

where

$$L = L_G(g) + L_M(g, \phi). \tag{II.2}$$

Let

$$A_G = \int dx L_G(g) \tag{II.3}$$

and

$$A_{GM} = \int dx L_M(g, \phi) \tag{II.4}$$

Both actions A_G and A_{GM} are assumed to be Lorentz invariant.

In order for the total action A to represent gravitational interaction [2], the action also has to be invariant with respect to the G gauge group (gravitational gauge group). Its infinitesimal generators (Lie algebra) consist of smooth vector fields, vanishing at infinity. This invariance is represented by smooth coordinate transformations, i.e., general covariance.

The action A_G represents gravitational selfinteraction. Its Lorentz invariance gives an energy tensor (translation invariant) and an energy-momentum tensor. The same is true for the action A_{GM} which represents the gravitational interaction with matter. The actions A_G and A_{GM} are in additional also G-gauge invariant.

In the next two sections we compute the energy tensors and energy-momentum tensors for these actions and the implications of the additional G-gauge symmetry.

III. Gravitational Selfinteraction

With the notations and the results in the appendices and form [3] and [4] the action for gravitational selfinteraction is given by

$$A_G = \int dx L_G(G) \tag{III.1}$$

where the Lagrangian is given by

$$L_G(G) = \sqrt{g}K + \partial_\mu B^\mu . \tag{III.2}$$

The divergence term is resopnsible for the action to be G-gauge invariant. Deleting the divergence results in a Lorentz invariant action only.

We now find the following quantities

$$H_0{}^{\alpha\beta,\mu} \equiv \frac{\partial(\sqrt{g}K)}{\partial g_{\alpha\beta}, \mu} \tag{III.3}$$

$$H_0{}^{\alpha\beta,\mu} = -\frac{1}{2} g^{\alpha\beta} B^\mu + \sqrt{g}\left[\Gamma^{\mu\alpha\beta} - \frac{1}{2}\left(g^{\mu\alpha}\Gamma^{\sigma\beta}{}_\sigma + g^{\mu\beta}\Gamma^{\sigma\alpha}{}_\sigma\right)\right] . \tag{III.4}$$

The Euler derivative is given by

$$G_0{}^{\alpha\beta} \equiv \varepsilon(g_{\alpha\beta})\sqrt{g}K \tag{III.5}$$

$$G_0{}^{\alpha\beta} = -\sqrt{g}\left[R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right] \tag{III.6}$$

The quantity in the bracket is also known as the Einstein tensor.

The energy tensor is given by

$$E_0{}^\mu{}_\sigma = H_o{}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} - \delta^\mu{}_\sigma \sqrt{g}K \tag{III.7}$$

Introducing

$$h^{\alpha\beta} = \sqrt{g} g^{\alpha\beta} \ , \ h_{\alpha\beta} = \frac{1}{\sqrt{g}} g_{\alpha\beta} \tag{III.8}$$

we get

$$E_0{}^\mu{}_\sigma = \Gamma^\beta{}_{\alpha\beta}\partial_\sigma h^{\alpha\beta} - \Gamma^\mu{}_{\alpha\beta}\partial_\sigma h^{\alpha\beta} - \delta^\mu{}_\sigma \sqrt{g}K \tag{III.9}$$

$E_0{}^\mu{}_\sigma$ is not G-gauge invariant and thus is only a Lorentz tensor.

From translation invariance we find

$$\partial_\mu E_0{}^\mu{}_\sigma = -G_0^{\alpha\beta} g_{\alpha\beta,\sigma} \tag{III.10}$$

Now we look at the other auxiliary quantities and find

$$\begin{aligned} P_0{}^{\mu\alpha}{}_\lambda &= 0 && \text{(III.11)} \\ K_0{}^{\mu\alpha}{}_\lambda &= 2H_0{}^{\alpha\beta,\mu} g_{\beta\lambda} && \text{(III.12)} \\ Z_0{}^\mu{}_\sigma &= 0 && \text{(III.13)} \end{aligned}$$

From G-gauge invariance one finds

$$\frac{1}{2}\left[K_0{}^{\alpha\beta}{}_\lambda + K_0{}^{\beta\alpha}{}_\lambda\right] = -\partial_\mu\left[\frac{\partial B^\mu}{\partial g_{\alpha\varrho,\beta}} g_{\varrho\lambda} + \frac{\partial B^\mu}{\partial g_{\beta\varrho,\alpha}} g_{\varrho\lambda}\right] \tag{III.14}$$

and from

$$\frac{\partial B^\mu}{\partial g_{\alpha\varrho,\beta}} g_{\varrho\lambda} = -\delta^\alpha{}_\lambda h^{\mu\beta} + \frac{1}{2}\delta^\beta{}_\lambda h^{\mu\alpha} + \frac{1}{2} g^\mu{}_\lambda h^{\alpha\beta} \tag{III.15}$$

we then get

$$\frac{1}{2}\left[K_0{}^{\alpha\beta}{}_\lambda + K_0{}^{\beta\alpha}{}_\lambda\right] = \partial_\mu\left[\frac{1}{2}\delta^\alpha{}_\lambda h^{\mu\beta} + \frac{1}{2}\delta^\beta{}_\lambda h^{\mu\alpha} - \delta^\mu{}_\lambda h^{\alpha\beta}\right]. \tag{III.16}$$

Observe now that

$$\partial_\alpha\partial_\beta K_0{}^{\alpha\beta}{}_\lambda = 0 \tag{III.17}$$

Also

$$W_0{}^{\lambda\mu\alpha} = \partial_\sigma\left[\eta^{\alpha\sigma} h^{\mu\lambda} + \eta^{\mu\sigma} h^{\alpha\lambda} - \eta^{\mu\alpha} h^{\mu\lambda} - \eta^{\sigma\lambda} h^{\mu\alpha}\right] \tag{III.18}$$

gives

$$t^{\mu\alpha} = \partial_\lambda\partial_\sigma\left[\eta^{\alpha\sigma} h^{\mu\lambda} + \eta^{\mu\sigma} h^{\alpha\lambda} - \eta^{\mu\alpha} h^{\sigma\lambda} - \eta^{\sigma\lambda} h^{\mu\alpha}\right] \tag{III.19}$$

Since $\partial_\alpha\partial_\beta K_0{}^{\alpha\beta}{}_\lambda = 0$ and $t^{\mu\alpha}$ is only a Lorentz tensor but not G-gauge invariant, the G-gauge invariant energy-momentum tensor is identically zero.

This also implies that a G-gauge invariant action, whose Lagrangian involves only up to first order derivatives of the fields and no divergence term, has energy-momentum tensor zero.

That the energy-momentum tensor vanishes identically is expressed by the identity

$$E_0^\mu{}_\sigma + 2G_0{}^{\mu\alpha} g_{\alpha\sigma} + \partial_\lambda K_0{}^{\lambda\mu}{}_\sigma \equiv 0 \tag{III.20}$$

Finally we get from G gauge invariance the identity

$$2\partial_\beta\left(G_0{}^{\beta\alpha} g_{\alpha\lambda}\right) = G_0{}^{\alpha\beta} g_{\alpha\beta,\lambda} \tag{III.21}$$

This is the Bianchi identity.

IV. Gravitational Interaction Term with Matter

The gravitational interaction term with matter is given by the action

$$A_{GM} = \int dx L_M(g, \phi)$$

This action is Lorentz invariant and also G-gauge invariant.

The Lagrangian

$$L_M(g, \phi) = L_M(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \phi, \partial_\sigma \phi) \tag{IV.1}$$

is assumed to depend only on the fields $g_{\alpha\beta}, \phi$ and on their first derivatives, and has no boundary term. Thus the energy-momentum tensor vanishes identically.

We now find the following quantities

$$H_M{}^{\alpha\beta,\mu} \equiv \frac{\partial L_M}{\partial g_{\alpha\beta,\mu}} \tag{IV.2}$$

This can only be evaluated if the matter fields ϕ are specified.

Similarly for

$$H_M{}^{\mu} \equiv \frac{\partial L_M}{\partial \phi_\mu} \tag{IV.3}$$

The Euler derivatives are given by

$$\varepsilon(g_{\alpha\beta}) L_M \equiv M^{\alpha\beta} \tag{IV.4}$$

We call $M^{\alpha\beta}$ the gravitational stress tensor.

$$G \equiv \varepsilon(\phi) L_M. \tag{IV.5}$$

is the Euler derivative of the matter fields. The energy tensor for the action A_{GM} is given by

$$E_1{}^{\mu}{}_{\sigma} = E_M{}^{\mu}{}_{\sigma} + H_M{}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} \tag{IV.6}$$

where

$$E_M^{\mu}{}_{\sigma} = H_M{}^{\mu}{}_{\sigma} \phi_\sigma - \delta^\mu{}_\sigma L_M \tag{IV.7}$$

is the energy tensor for the matter field. Again, both ${E_1}^\mu{}_\sigma$ and ${E_M}^\mu{}_\sigma$ are only Lorentz tensors, because in general they are not G-gauge invariant.

From translation invariance we find

$$\partial_\mu {E_1}^\mu{}_\sigma = -G\phi_\sigma - M^{\alpha\beta} g_{\alpha\beta,\sigma} \tag{IV.8}$$

Now we look at the other auxiliary quantities and find

$${K_1}^{\mu\alpha}{}_\lambda = H^\mu S^\alpha_\lambda + 2{H_M}^{\alpha\beta,\mu} g_{\beta\lambda} \tag{IV.9}$$

and

$${Z_1}^\mu{}_\sigma = {E_1}^\mu{}_\sigma + GS^\mu_\sigma + 2M^{\mu\beta} g_{\beta\sigma} + \partial_\lambda {K_1}^{\lambda\mu}{}_\sigma \tag{IV.10}$$

Since ${Z_1}^\mu{}_\sigma$ vanishes identically we obtain

$${E_M}^\mu{}_\sigma + GS^\mu_\sigma + \partial_\lambda(H^\lambda S^\mu{}_\sigma) + 2M^{\mu\beta} g_{\beta\sigma} + {H_M}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} + 2\partial_\lambda \left[{H_M}^{\mu\beta,\lambda} g_{\beta\sigma} \right] = 0 \tag{IV.11}$$

From G-gauge invariance we finally get the identity

$$\partial_\beta \left[GS^\beta{}_\lambda + 2M^{\beta\alpha} g_{\alpha\lambda} \right] = G\phi_\lambda + M^{\alpha\beta} g_{\alpha\beta,\lambda} \tag{IV.12}$$

V. General Gravitational Interaction

We now look at the total action

$$A = A_G + A_{GM} \tag{V.1}$$

with the corresponding Lagrangian

$$L = L_G + L_M \tag{V.2}$$

The auxiliary quantities then read

$$H^{\alpha\beta,\mu} \equiv \frac{\partial[\sqrt{g}K + L_M]}{\partial g_{\alpha\beta,\mu}} \tag{V.3}$$

$$H^{\alpha\beta,\mu} = {H_0}^{\alpha\beta,\mu} + {H_M}^{\alpha\beta,\mu} \tag{V.4}$$

$$H^\mu \equiv \frac{\partial L_M}{\partial \phi_\mu} \tag{V.5}$$

$$H^\mu = {H_M}^\mu \tag{V.6}$$

For the Euler derivatives we get

$$G^{\alpha\beta} \equiv \varepsilon(g_{\alpha\beta})[\sqrt{g}K + L_M] \tag{V.7}$$

$$G^{\alpha\beta} = G_0{}^{\alpha\beta} + M^{\alpha\beta} \tag{V.8}$$

$$G \equiv \varepsilon(\phi)L_M \tag{V.9}$$

The total energy tensor then becomes

$$E^{\mu}_{\ \sigma} = E_0{}^{\mu}{}_{\sigma} + E_M{}^{\mu}{}_{\sigma} + H_M{}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} \tag{V.10}$$

where $E_0{}^{\mu}{}_{\sigma}$ is given by (III.9) and $E_M{}^{\mu}{}_{\sigma}$ by (IV.7).

$E^{\mu}_{\ \sigma}$ is only a Lorentz tensor.

From translation invariance we get

$$\partial_\mu E^{\mu}_{\ \sigma} = -G^{\alpha\beta} g_{\alpha\beta,\sigma} - G\phi_\sigma \tag{V.11}$$

The G-gauge invariant energy-momentum tensors belonging to the actions A_G and A_{GM} are both identically zero. Thus the overall G-gauge invariant energy-momentum tensor is zero.

This is reflected in the following two identities

$$E_0{}^{\mu}{}_{\sigma} + 2G_0{}^{\mu\alpha} g_{\alpha\sigma} \partial_\lambda K_0{}^{\lambda\mu}{}_{\sigma} = 0 \tag{V.12}$$

$$E_M{}^{\mu}{}_{\sigma} + GS^{\mu}_{\ \sigma} + \partial_\lambda (H^\lambda S^{\mu}_{\ \sigma}) + 2M^{\mu\beta} g_{\beta\sigma} + H_M{}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} + 2\partial_\lambda \left[H_M{}^{\mu\beta,\lambda} g_{\beta\sigma} \right] = 0 \tag{V.13}$$

Finally G-gauge invariance gives the two identities

$$2\partial_\beta \left(G_0{}^{\beta\alpha} g_{\alpha\lambda} \right) = G_0{}^{\alpha\beta} g_{\alpha\beta,\lambda} \tag{V.14}$$

$$\partial_\beta \left[GS^{\beta}_{\ \lambda} + 2M^{\beta\alpha} g_{\alpha\lambda} \right] = G\phi_\lambda + M^{\alpha\beta} g_{\alpha\beta,\lambda} \tag{V.15}$$

Now we assume the equations of motion to be satisfied, i.e.,

$$G = 0 \tag{V.16}$$

$$G^{\alpha\beta} = 0 \tag{V.17}$$

$G = 0$ is the equation of motion for the matterfield and $G^{\alpha\beta} = 0$ is the Einstein equation

$$\sqrt{g} \left[R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right] = M^{\alpha\beta} \tag{V.18}$$

From translation invariance we get the conservation law

$$\partial_\mu E^\mu{}_\sigma = 0 \tag{V.19}$$

This means that the total energy is conserved. This statement is reflected in the equations

$$\begin{aligned} \partial_\mu {E_0}^\mu{}_\sigma &= M^{\alpha\beta} g_{\alpha\beta,\sigma} && \text{(V.20)} \\ \partial_\mu {E_M}^\mu{}_\sigma &= -M^{\alpha\beta} g_{\alpha\beta,\sigma} - \partial_\mu \left[{H_M}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} \right] && \text{(V.21)} \end{aligned}$$

Finally we have the energy-balance equation

$$E^\mu{}_\sigma + \partial_\lambda \left[{K_0}^{\lambda\mu}{}_\sigma + {K_1}^{\lambda\mu}{}_\sigma \right] = 0 \tag{V.22}$$

which is reflected in the equations

$$ {E_0}^\mu{}_\sigma - 2M^{\mu\beta} g_{\alpha\sigma} + \partial_\lambda {K_0}^{\lambda\mu}{}_\sigma = 0 \tag{V.23}$$

$$ {E_M}^\mu{}_\sigma + 2M^{\mu\beta} g_{\beta\sigma} + {H_M}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} + \partial_\lambda \left[H^\lambda S^\mu{}_\sigma + 2{H_M}^{\mu\beta,\lambda} g_{\beta\sigma} \right] = 0 \tag{V.24}$$

and from G-gauge invariance the relation

$$2\partial_\beta \left[M^{\beta\alpha} g_{\alpha\lambda} \right] = M^{\alpha\beta} g_{\alpha\beta,\lambda} \tag{V.25}$$

VI. Example: Gravitational Interaction with a Massive Vectorfield

The gravitational interaction term for this model is given by the Lagrangian

$$L_M(g, \phi_\alpha) = \sqrt{g}[g^{\alpha\beta} g^{\mu\nu} (D_\mu \phi_\alpha)(D_\nu \phi_\beta) - m^2 g^{\alpha\beta} \phi_\alpha \phi_\beta] \tag{VI.1}$$

with the covariant derivative

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - \Gamma^\sigma{}_{\alpha\mu} \phi_\sigma \tag{VI.2}$$

for the vectorfield $\{\phi_\alpha\}$.

We first compute the auxiliary quantities

$$H_M{}^{\alpha,\mu} \equiv \frac{\partial L_M}{\partial \phi_{\alpha,\mu}} \quad \text{(VI.3)}$$

$$H_M{}^{\alpha\beta,\mu} \equiv \frac{\partial L_M}{\partial g_{\alpha\beta,\mu}} \quad \text{(VI.4)}$$

$$H_M{}^{\alpha,\mu} = 2\sqrt{g} D^\mu \phi^\alpha \quad \text{(VI.5)}$$

$$H_M{}^{\alpha\beta,\mu} = -\frac{1}{2}\sqrt{g}[\phi^\alpha(D^\beta\phi^\mu + D^\mu\phi^\beta) + \phi^\beta(D^\alpha\phi^\mu + D^\mu\phi^\alpha) - \phi^\mu(D^\alpha\phi^\beta + D^\beta\phi^\alpha)] \quad \text{(VI.6)}$$

We have raised the indices with $g^{\alpha\beta}$.

For the Euler derivatives

$$G^\alpha \equiv \varepsilon(\phi_\alpha) L_M \quad \text{(VI.7)}$$

$$M^{\alpha\beta} \equiv \varepsilon(g_{\alpha\beta}) L_M \quad \text{(VI.8)}$$

we get

$$G^\alpha = -2\sqrt{g}[D_\mu D^\mu \phi^\alpha + m^2\phi^\alpha] \quad \text{(VI.9)}$$

$$M^{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}L_M - \sqrt{g}[(D^\mu\phi^\alpha)(D_\mu\phi^\beta) + (D^\alpha\phi^\mu)(D^\beta\phi_\mu) - m^2\phi^\alpha\phi^\beta] \quad \text{(VI.10)}$$
$$+ \frac{1}{2}\sqrt{g}D_\mu\left[\phi^\alpha(D^\beta\phi^\mu + D^\mu\phi^\beta) + \phi^\beta(D^\alpha\phi^\mu + D^\mu\phi^\alpha) - \phi^\mu(D^\alpha\phi^\beta + D^\beta\phi^\alpha)\right]$$

The energy tensor for the matterfield alone is given by

$$E_M{}^\mu{}_\sigma = 2\sqrt{g}(D^\mu\phi^\alpha)\phi_{\alpha,\sigma} - \delta^\mu{}_\sigma L_M \quad \text{(VI.11)}$$

The total energy tensor for this gravitational interaction then becomes

$$E^\mu{}_\sigma = E_0{}^\mu{}_\sigma + E_M{}^\mu{}_\sigma + H_M{}^{\alpha\beta,\mu} g_{\alpha\beta,\sigma} \quad \text{(VI.12)}$$

where $E_0{}^\mu{}_\sigma$ is given by (III.9) as

$$E_0{}^\mu{}_\sigma = \Gamma^\beta{}_{\alpha\beta}\partial_\sigma h^{\alpha\mu} - \Gamma^\mu{}_{\alpha\beta}\partial_\sigma h^{\alpha\beta} - \delta^\mu{}_\sigma\sqrt{g}K \quad \text{(VI.13)}$$

The equations of motion now read

$$D_\mu D^\mu \phi_\alpha + m^2\phi_\alpha = 0 \quad \text{(VI.14)}$$

$$
\begin{aligned}
R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R &= \frac{1}{2}g^{\alpha\beta}\left[(D_\mu\phi_\sigma)(D^\mu\phi^\sigma) - m^2\phi_\sigma\phi^\sigma\right] \\
&\quad -(D^\mu\phi^\alpha)(D_\mu\phi^\beta) - (D^\alpha\phi^\mu)(D^\beta\phi_\mu) \\
&\quad +m^2\phi^\alpha\phi^\beta + \frac{1}{2}D_\mu[\phi^\alpha(D^\beta\phi^\mu + D^\mu\phi^\beta) + \phi^\beta(D^\alpha\phi^\mu + D^\mu\phi^\alpha) \\
&\quad -\phi^\mu(D^\alpha\phi^\beta + D^\beta\phi^\alpha)] \qquad \text{(VI.15)}
\end{aligned}
$$

In addition we have the conservation law

$$
\partial_\mu E^\mu{}_\sigma = 0 \qquad \text{(VI.16)}
$$

where $E^\mu{}_\sigma$ is given by (VI.12).

For the gravitational stress tensor we also have the equation

$$
2\partial_\beta\left[M^{\beta\alpha}g_{\alpha\lambda}\right] = M^{\alpha\beta}g_{\alpha\beta,\lambda} \qquad \text{(VI.17)}
$$

which is equivalent to the Bianchi identity.

VII. Conclusions

We consider Lorentz-invariance as the fundamental symmetry of all of physics. Within the framework of the Lagrange Formalism a Lorentz-invariant action gives raise to the Energy tensor (due to translation invariance) and to the Energy-momentum tensor (due to proper Lorentz-rotations). If the equations of motion are satisfied both these tensors are conserved. For gravitational interactions there is the additional symmetry of G-gauge transformations vanishing at infinity. If the action if G-gauge invariant, the G-gauge covariant energy-momentum tensor vanishes identically. As we will see in a forthcoming paper this statement will give a precise meaning of the folklore statement "The right hand side of Einstein's equations is given by the energy-momentum tensor of matter." The energy tensor for the gravitational interaction is however not G-gauge covariant. But if the equations of motion are satisfied, the energy tensor is conserved.

We hope that this paper puts to rest all the speculative statements about the energy-momentum tensor in General Relativity.

Appendix A: Notations

Let $\eta_{\mu\nu}$ denote the Lorentz metric tensor with signature $(+,-,-,-)$ and $\eta^{\mu\nu}$ its inverse. The Lorentz 4-volume element is represented by dx. We use the abbreviation $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and for any field ϕ on Minkowski space $\partial_\mu \phi = \phi_\mu$. Let $g_{\alpha\beta}$ be a symmetric Lorentz tensor field on Minkowski space and $g^{\alpha\beta}$ be its inverse, i.e.,

$$g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha{}_\beta$$

We introduce the following quantities

$$\begin{aligned}
g &= -Det(g_{\mu\nu}) \\
\Gamma_{\mu\alpha\beta} &= \frac{1}{2}[g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}] \\
\Gamma^\mu{}_{\alpha\beta} &= g^{\mu\sigma}\Gamma_{\sigma\alpha\beta} \\
\Gamma^{\mu\sigma}{}_\beta &= g^{\sigma\alpha}\Gamma^\mu{}_{\alpha\beta} \\
\Gamma^{\mu\sigma\varrho} &= g^{\varrho\beta}\Gamma^{\mu\sigma}{}_\beta \\
K_{\mu\nu} &= \Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta} \\
B^\mu &= \sqrt{g}[\Gamma^{\mu\alpha}{}_\alpha - \Gamma^{\alpha\mu}{}_\alpha] \\
R_{\mu\nu} &= \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\alpha\mu,\nu} - K_{\mu\nu} \\
R^\alpha{}_\nu &= g^{\alpha\mu}R_{\mu\nu} \\
R^{\alpha\beta} &= g^{\beta\nu}R^\alpha{}_\nu \\
R &= g^{\mu\nu}R_{\mu\nu} \\
K^{\alpha\beta} &= g^{\alpha\mu}g^{\beta\nu}K_{\mu\nu} \\
K &= g^{\mu\nu}K_{\mu\nu} \\
H_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\
H^\alpha{}_\nu &= g^{\alpha\mu}H_{\mu\nu} \\
H^{\alpha\beta} &= g^{\beta\nu}H^\alpha{}_\nu
\end{aligned}$$

We then have the following relations

$$R\sqrt{g} = K\sqrt{g} + \partial_\mu B^\mu \tag{1}$$

$$\varepsilon(g_{\alpha\beta})(K\sqrt{g}) \equiv \left[\frac{\partial}{\partial g_{\alpha\beta}} - \partial_\mu \frac{\partial}{\partial g_{\alpha\beta,\mu}}\right] \quad (2)$$

$$\partial_\mu(\sqrt{g}H^\mu{}_\sigma) = \frac{1}{2}\sqrt{g}H^{\alpha\beta}g_{\alpha\beta,\sigma} \quad (3)$$

Appendix B: Calculus of Variation for a Lorentz-invariant Action

We collect here the resuts in [3].

Let ϕ represent several multicomponent fields on Minkowski space. Summation over Lorentz-, spinor-, and internal indices is always implied. The action is given by

$$A = \int dx L_0 + \int dx \partial_\mu B^\mu$$

where

$$L_0 = L_0(\phi, \partial_\mu\phi), \quad B^\mu = B^\mu(\phi, \partial_\alpha\phi),$$

and is assumed to be Lorentz invariant.

The infinitesimal field variation (local variation) is defined by

$$(\delta_*\phi)(x) = \bar{\phi}(x) - \phi(x).$$

δ_* commutes with the derivative, i.e.,

$$\delta_*\partial_\mu = \partial_\mu\delta_*.$$

An infinitesimal coordinate transformation on x^μ results in new coordinates $\bar{x}^\mu$. The coordinate variation is then defined by

$$\delta x^\mu = \bar{x}^\mu - x^\mu.$$

The variation of a field, induced by a coordinate variation, is given by

$$(\delta\phi)(x) = \bar{\phi}(\bar{x}) - \phi(x).$$

We then have the relation

$$\delta_* = \delta - (\delta x^\mu)\partial_\mu$$

as applied to any field.

The variational principle now reads

$$\delta A = \int dx\, \delta_* L_0 + \int dx \partial_\mu [L_0 \delta x^\mu + (\partial_\alpha B^\alpha)\delta x^\mu + \delta_* B^\mu]$$

We now introduce the following abbreviations

$$\begin{aligned} H^\mu &\equiv \frac{\partial L_0}{\partial \phi_\mu} && (1) \\ G &\equiv \varepsilon(\phi) L_0 \equiv \frac{\partial L_0}{\partial \phi} - \partial_\mu H^\mu \quad \text{Euler derivative} && (2) \\ E^\mu{}_\sigma &\equiv H^\mu \phi_\sigma - \delta^\mu{}_\sigma L_0 \quad \text{Energy tensor} && (3) \end{aligned}$$

Then the variational principle becomes

$$\begin{aligned} \delta A &= \int dx [G\delta\phi - G\phi_\sigma \delta x^\sigma] \\ &+ \int dx \partial_\mu [-E^\mu{}_\sigma \delta x^\sigma + H^\mu \delta\phi + B^\mu(\partial_\alpha \delta x^\alpha) - B^\alpha(\partial_\alpha \delta x^\mu) + \delta B^\mu] \end{aligned}$$

For coordinate variations with

$$\delta\phi = -S^\beta{}_\lambda(\phi)\partial_\beta \delta x^\lambda$$

and the abbreviations

$$\begin{aligned} P^{\mu\beta}{}_\lambda &\equiv \delta^\mu{}_\lambda B^\beta - \delta^\beta{}_\lambda B^\mu + \frac{\partial B^\mu}{\partial \phi} S^\beta{}_\lambda + \frac{\partial B^\mu}{\partial \phi_\beta}\phi_\lambda + \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta{}_\lambda && (4) \\ K^{\mu\alpha}{}_\lambda &\equiv H^\mu S^\alpha{}_\lambda + P^{\mu\alpha}{}_\lambda && (5) \end{aligned}$$

The variational principle finally reads

$$\begin{aligned} \delta A &= \int dx \left[-G\phi_\sigma \delta x^\sigma - G S^\beta{}_\lambda \partial_\beta \delta x^\lambda\right] \\ &+ \int dx \partial_\mu \left[-E^\mu{}_\sigma \delta x^\sigma - K^{\mu\beta}{}_\lambda \partial_\beta \delta x^\lambda - \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta{}_\lambda \partial_\alpha \partial_\beta \delta x^\lambda\right] \end{aligned}$$

or

$$\begin{aligned} \delta A &= \int dx \left[\partial_\beta (G S^\beta{}_\lambda) - G\phi_\lambda\right] \delta x^\lambda \\ &+ \int dx \partial_\mu \left[-\{E^\mu{}_\sigma + G S^\mu{}_\sigma\}\delta x^\lambda - K^{\mu\beta}{}_\lambda \partial_\beta \delta x^\lambda - \frac{\partial B^\mu}{\partial \phi_\alpha} S^\beta{}_\lambda \partial_\alpha \partial_\beta \delta x^\lambda\right] \end{aligned}$$

Translation invariant implies

$$\partial_\mu E^\mu{}_\sigma + G\phi_\sigma = 0 \tag{I}$$

Now raising and lowering indices will be done with the Lorentz metric.

We finally introduce the abbreviations

$$\begin{aligned} Z^\mu{}_\sigma &\equiv E^\mu{}_\sigma + GS^\mu{}_\sigma + \partial_\lambda K^{\lambda\mu}{}_\sigma & (6) \\ W^{\lambda\mu\alpha} &\equiv \frac{1}{2}\left(K^{\mu\alpha\lambda} + K^{\alpha\mu\lambda}\right) - \frac{1}{2}\left(K^{\lambda\mu\alpha} + K^{\mu\lambda\alpha}\right) - \frac{1}{2}\left(K^{\alpha\lambda\mu} + K^{\lambda\alpha\mu}\right) & (7) \end{aligned}$$

Proper Lorentz invariance together with translation invariance now gives

$$Z^{\mu\alpha} = Z^{\alpha\mu} \tag{II}$$

$$\partial_\mu Z^\mu{}_\sigma = -G\phi_\sigma + \partial_\mu(GS^\mu{}_\sigma) + \partial_\mu\partial_\lambda K^{\lambda\mu}{}_\sigma \tag{III}$$

The energy-momentum tensor is given as follows: Let

$$t^{\mu\alpha} \equiv \partial_\lambda W^{\lambda\mu\alpha}.$$

Then

$$t^{\mu\alpha} = t^{\alpha\mu}$$

(i) If $\partial_\mu\partial_\lambda K^{\lambda\mu\alpha} = 0$, then

$$T^{\mu\alpha} = Z^{\mu\alpha}$$

and

$$\partial_\mu t^{\mu\alpha} = 0$$

(ii) If $\partial_\mu\partial_\lambda K^{\lambda\mu\alpha} \neq 0$, then

$$T^{\mu\alpha} = Z^{\mu\alpha} + t^{\mu\alpha}$$

The energy-momentum tensor is symmetric and is conserved provided the equations of motion are satisfied.

Appendix C: Raising and Lowering Operators

$$h^{\alpha\beta} = \sqrt{g} g^{\alpha\beta}, h_{\alpha\beta} = \frac{1}{\sqrt{g}} g_{\alpha\beta}$$

Many expressions in the theory of gravity are simpler if written in terms of the above quantities. Here raising and and lowering is performed by $h^{\alpha\beta}$ and $h_{\alpha\beta}$ respectively.

We introduce the abbreviations

$$\begin{aligned} A^{\nu}{}_{\mu\lambda} &\equiv h_{\mu\beta}\partial_\lambda h^{\beta\nu} = -h^{\nu\beta}\partial_\lambda h_{\beta\mu} \\ A_\lambda &\equiv A^{\mu}{}_{\mu\lambda} \end{aligned}$$

Then

$$\begin{aligned} A^{\nu}{}_{\mu\lambda} &= \delta^{\nu}{}_{\mu}\Gamma^{\sigma}{}_{\sigma\lambda} - g^{\nu\beta}(\Gamma_{\beta\mu\lambda} + \Gamma_{\mu\beta\lambda}) \\ A_\lambda &= 2\Gamma^{\sigma}{}_{\sigma\lambda} \quad , \quad \partial_\mu A_\nu = \partial_\nu A_\mu \\ A^{\nu\sigma}{}_{\lambda} &= \partial_\lambda h^{\nu\sigma} \\ A_{\nu\sigma\lambda} &= -\partial_\lambda h_{\nu\sigma} \end{aligned}$$

We also have the representation

$$\begin{aligned} \Gamma^{\mu}{}_{\nu\sigma} &= \frac{1}{4}\{\delta^{\mu}{}_{\nu}A_\sigma + \delta^{\mu}{}_{\sigma}A_\nu - h_{\nu\sigma}A^\mu\} \\ &- \frac{1}{2}\{A^{\mu}{}_{\nu\sigma} + A^{\mu}{}_{\sigma\nu} - h^{\mu\alpha}A_{\nu\sigma\alpha}\} \\ K\sqrt{g} &= \frac{1}{8}A^\alpha A_\alpha + \frac{1}{4}\left[2A^{\mu\alpha}{}_{\sigma}A^{\sigma}_{\alpha\mu} - h^{\alpha\nu}A^{\mu\sigma}{}_{\alpha}A_{\sigma\mu\nu}\right] \\ B^\mu &= -A^{\mu\alpha}{}_{\alpha} - \frac{1}{2}A^\mu \end{aligned}$$

References

1. W. Wyss, "Gauge Invariant Electromagnetic Interactions" (to be published).

2. W. Wyss, *Helv. Phys. Acta*, **38**, 469 (1965).
3. W. Wyss, "The Energy-Momentum Tensor in Classical Field Theory" (to be published).
4. M. Carmeli, "Classical Fields: General Relativity and Gauge Theory," John Wiley and Sons, 1982.